11.10/11.11.

REMAINDERS FROM TAYLOR SERIES

$$f(x) \approx T(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$T_n(x) = \sum_{i=0}^{n} c_i (x-a)^i$$

TAYLOR POLYNOMIAL
OF DEGREE n

IF WE USE $T_n(x)$ TO APPROXIMATE $f(x)$
HOW FAR COULD $T_n(x)$ BE FROM $f(x)$?

$$R_n(x) = f(x) - T_n(x) \longrightarrow f(x) = T_n(x) + R_n(x)$$

REMAINDER/ERROR FROM USING T_n TO APPROXIMATE f

$$T(x) = \sum_{i=0}^{\infty} c_i (x-a)^i$$

$$T_n(x) = \sum_{i=0}^{n} c_i (x-a)^i$$

$$R_n(x) = T(x) - T_n(x) = \sum_{i=n+1}^{\infty} c_i (x-a)^i$$

SINCE WE USE $T_n(x)$ TO APPROXIMATE $f(x)$ WHEN WE DON'T KNOW $f(x)$

THEREFORE WE DON'T KNOW $R_n(x)$

CONSIDER ALTERNATING SERIES

$$S = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$

$a_n > 0$

$\lim_{n\to\infty} a_n = 0$

a_n DECREASING

$$S_n = a_1 - a_2 + a_3 - a_4 \cdots (-1)^{n+1} a_n$$

$$R_n = S - S_n = (-1)^{n+2} a_{n+1} + (-1)^{n+3} a_{n+2} + (-1)^{n+4} a_{n+3}$$

ASSUME n IS EVEN

$$S_n = a_1 - a_2 + a_3 - a_4 \cdots + a_{n-1} - a_n$$

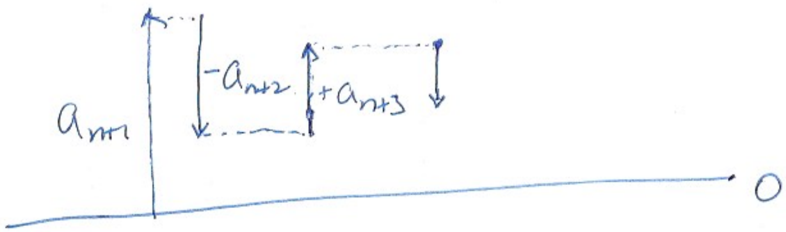
$$0 < R_n = a_{n+1} - a_{n+2} + a_{n+3} - \cdots < a_{n+1}$$

$$0 \le R = S - S_n < a_{n+1}$$

$$0 < S - S_n < a_{n+1}$$

$$\underbrace{S_n < S < S_n + |a_{n+1}|}$$

?



eg. CONSIDER $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{12 - 6 + 4 - 3}{12} = \frac{7}{12}$$

$$\frac{7}{12} < S < \frac{7}{12} + \left|\frac{1}{5}\right|$$

$$\frac{7}{12} < S < \frac{47}{60}$$

FOR n ODD

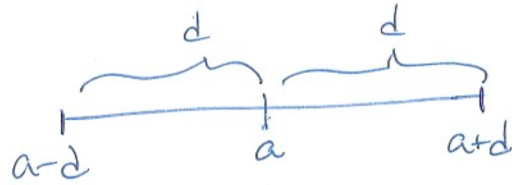
$$S_n - a_{n+1} < S < S_n$$

IF YOU USE $T_n(x)$ TO APPROXIMATE $f(x)$

IF $|f^{(n+1)}(x)| \le M$ FOR $|x-a| \le d$

THEN $|R_n(x)| = |f(x) - T_n(x)| \le \dfrac{M}{(n+1)!}|x-a|^{n+1}$

FOR $|x-a| \le d$



CENTER OF T.S.

FIND A MAX ABS VALUE OF $f^{(n+1)}$ ON THAT INTERVAL TO GET M

A VALUE THAT IS GUARANTEED TO BE LARGER THAN $f^{(n+1)}$

APPROXIMATE $\sin 1$ USING $T_5(x)$ AND FIND AN INTERVAL FOR $\sin 1$.

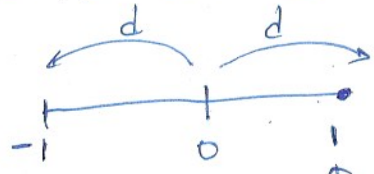
$f(x) = \sin x$ $\qquad \sin 1 \approx T_5(1) = 1 - \frac{1}{3!} + \frac{1}{5!} = 1 - \frac{1}{6} + \frac{1}{120} = \frac{120 - 20 + 1}{120}$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \qquad\qquad = \frac{101}{120} = 0.841\overline{6}$$

NEED TO ESTIMATE $R_5(1)$

① $|R_5(x)| \le \frac{M}{6!}|x - 0|^6$ $\qquad$ WHERE $|f^{(6)}(x)| \le M$

FOR $|x - 0| \le d = 1$



WE WANT TO LET $x = 1$ IN T.S.

$|f^{(6)}(x)| = |-\sin x| \le \frac{\sqrt{3}}{2}$

FOR $|x - 0| \le 1$

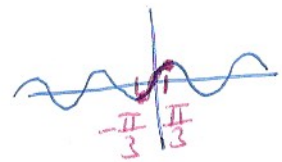
USE $M = \frac{\sqrt{3}}{2}$

$f^{(6)}(x) = -\sin x$

$|-\sin x| \le M$ ON $[-1, 1]$

$$|R_5(x)| \le \frac{\frac{\sqrt{3}}{2}}{6!} x^6$$

$$|R_5(1)| \le \frac{\frac{\sqrt{3}}{2}}{6!} 1^6 = \frac{\sqrt{3}}{2 \cdot 6!}$$



$\frac{\pi}{3} \approx 1.05$

SO $\frac{101}{120} - \frac{\sqrt{3}}{2 \cdot 6!} < \sin 1 < \frac{101}{120} + \frac{\sqrt{3}}{2 \cdot 6!}$

$0.8404638536 < \sin 1 < 0.8428694797$

ON $[-\frac{\pi}{3}, \frac{\pi}{3}]$

$|\sin x| \le \frac{\sqrt{3}}{2}$

ALSO TRUE ON $[-1, 1]$

(WHICH IS INSIDE $[-\frac{\pi}{3}, \frac{\pi}{3}]$)

IF YOU WANT TO APPROXIMATE $\sin 1$ WITHIN 10^{-7}

$$|R_n(x)| \leq 10^{-7}$$

$$|R_n(x)| \leq \frac{M}{(n+1)!} x^{n+1} \quad \text{ON } [-1,1]$$

SO IF WE MAKE $\left| \frac{M}{(n+1)!} x^{n+1} \right| \leq 10^{-7}$ ON $[-1,1]$

$|x^{n+1}| \leq 1$
ON $[-1,1]$

$$\frac{\frac{\sqrt{3}}{2}}{(n+1)!} \leq 10^{-7}$$

$$\underbrace{10^{7} \cdot \frac{\sqrt{3}}{2}}_{11,547,005} \leq (n+1)!$$

$$\boxed{n=10}$$

NEED $T_{10}(1) = \boxed{T_9(1)}$